On the aquitard–aquifer interface flow and the drawdown sensitivity with a partially penetrating pumping well in an anisotropic leaky confined aquifer

Qinggao Feng a, Hongbin Zhan b,∗

a Department of Geotechnical Engineering, Tongji University, Shanghai 200092, PR China
b Department of Geology and Geophysics, Texas A&M University, College Station, TX 77843-3115, USA

A R T I C L E   I N F O

Article history:
Received 3 September 2014
Received in revised form 19 November 2014
Accepted 21 November 2014
Available online 1 December 2014
This manuscript was handled by Corrado Corradini, Editor-in-Chief, with the assistance of Christophe Darnault, Associate Editor

Keywords:
Leaky confined aquifer
Leakage rate
Leakage volume
Partially penetrating well
Wellbore storage
Anisotropy

S U M M A R Y

A mathematical model for describing groundwater flow to a partially penetrating pumping well of a finite diameter in an anisotropic leaky confined aquifer is developed. The model accounts for the jointed effects of aquitard storage, aquifer anisotropy, and wellbore storage by treating the aquitard leakage as a boundary condition at the aquitard–aquifer interface rather than a volumetric source/sink term in the governing equation, which has never developed before. A new semi-analytical solution for the model is obtained by the Laplace transform in conjunction with separation of variables. Specific attention was paid on the flow across the aquitard–aquifer interface, which is of concern if aquitard and aquifer have different pore water chemistry. Moreover, Laplace-domain and steady-state solutions are obtained to calculate the rate and volume of (total) leakage through the aquitard–aquifer interface due to pump in a partially penetrating well, which is also useful for engineers to manager water resources. The sensitivity analyses for the drawdown illustrate that the drawdown is most sensitive to the well partial penetration. It is apparently sensitive to the aquifer anisotropic ratio over the entire time of pumping. It is moderately sensitive to the aquitard/aquifer specific storage ratio at the intermediate times only. It is moderately sensitive to the aquitard/aquifer vertical hydraulic conductivity ratio and the aquitard/aquifer thickness ratio with the identical influence at late times.

1. Introduction

The classical theory for groundwater flow to a partially penetrating well with a constant rate test in a leaky confined aquifer was first proposed by Hantush (1957) and later by others such as Hantush (1964), Halepaska (1972), Hunt (2005) and Perina and Lee (2006). Hantush (1957) assumed that the leakage from the aquitard was treated as a volumetric source/sink term contained in the governing flow equation which was later called “the Hantush approximation” (Zhan and Bian, 2006). This approximation was utilized in many studies including Hantush (1960, 1964), Lai and Su (1974), Zhan and Park (2003), Zhan and Bian (2006), Hunt and Scott (2007) and Yang and Yeh (2009). In reality, the aquitard leakage does not happen over the entire aquifer volume; rather, it only occurs at the aquitard–aquifer interface, thus should be treated as an interface problem, not a volumetric source/sink term.

Hantush (1967) realized the drawbacks of using the volumetric source/sink approximation and solved the equation governing flow toward a partially penetrating well in a leaky confined aquifer by regarding the leakage term as a boundary condition. However, Hantush (1967) neglected both the aquitard storage and the aquifer anisotropy, apparently for the sake of mathematical simplicity. Halepaska (1972) obtained the drawdown distribution for flow to a partially penetrating well of an infinitesimal radius in a thick, homogeneous, isotropic leaky confined aquifer, and neglected the aquitard storage as well. Hunt (2005) made some closed-form analytical solutions available for describing drawdown distribution in a leaky confined aquifer with a relatively thin aquitard. In the analysis of Hunt (2005), the well was treated as the superposition of many point sources and the aquitard compressibility was excluded. Hence, the solution obtained by Hunt (2005) cannot be used to analyze the effect of aquitard storage and wellbore storage. Sun and Zhan (2006) analyzed the flow to a horizontal well in a leaky confined aquifer by treating the aquitard leakage as occurred at the aquitard–aquifer interface, and they found that the Hantush approximation could result in non-negligible errors, especially at
The effects of aquifer anisotropy and wellbore storage on the drawdown distribution have been extensively investigated in studies such as Papadopoulos and Cooper (1967), Lai and Su (1974), Bear (1979), Moench (1985, 1997), Novakowski (1989), Ruud and Kabala (1997), Cassiani and Kabala (1998), Hemker (1999a, 1999b), Zhan et al. (2001), Park and Zhan (2002), Chang and Chen (2003), Yang and Yeh (2005), Mathias and Butler (2007), Pasandi et al. (2008), Chen et al. (2009), Yang and Yeh (2009), Fen and Yeh (2012) and Mishra et al. (2012). These studies showed that difference for drawdown would occur if the effect of aquifer anisotropy or wellbore storage was excluded, especially at early times.

Another interesting problem related to flow to a pumping well in a leaky confined aquifer was the induced flow at the aquitard–aquifer interface, which received increasing attention, particularly in groundwater resources management. Hantush (1964) obtained the rate and total volume of pumping induced leakage in an aquitard–aquifer system. Butler and Tsou (2003) presented a new approach to calculate the pumping-induced leakage at the aquitard–aquifer interface for infinite systems or laterally bounded aquifers. Zhan and Bian (2006) developed analytical or semi-analytical expressions to obtain the leakage rate and leakage volume at the aquitard–aquifer interface based on the solution of Hantush and Jacob (1955) and Hantush (1964) which the aquitard storage effect is not taken into consideration. Zhou et al. (2009) obtained (semi-)analytical solutions to analyze the rate and volume of leakage due to inject at a fully penetrating well in a leaky confined aquifer with considering aquitard storage.

After a careful literature review, it appears that none of the available studies have considered the combined effects of well partially penetration, aquitard storage, aquifer anisotropy and wellbore storage with a specific treatment of the aquitard leakage as an aquitard–aquifer interface flow problem under leaky confined aquifer condition. The aim of this study is to develop a general mathematical model for the flow to a partially penetrating constant-rate pumping well in an anisotropic leaky confined aquifer where the influence of the aquitard storage and wellbore storage are considered. Especially, the leakage is not regarded as a volumetric source/sink, but as an aquitard–aquifer interface problem. And more importantly, closed-form analytical and semi-analytical solutions are provided to calculate the rate and volume of leakage over any given time interval. With the help of these newly developed solutions, one can specifically delineate the importance of flow at the aquitard–aquifer interface. Additionally, a sensitivity analysis is made to study the degree of sensitivity of the drawdown to the parameters of the aquitard and aquifer.

2. Mathematical model and solution

Fig. 1 shows a partially penetrating pumping well in a leaky confined aquifer system. The main aquifer is homogeneous, anisotropic and of infinite lateral extent with constant thickness, which exists between an aquitard and an impermeable formation. The water supply from the upper unconfined aquifer or the surface water body above the aquitard is assumed to be sufficient, which can be regard as a constant head boundary. The aquitard storage is non-negligible. The pumping well of a finite radius is partially penetrating and the flow rate is assumed to be constant along the well screen. Neuman and Witherspoon (1972) confirm that the aquitard storage must be considered when one deals with leaky aquifer systems. The aquitard storage is considered in this model. Additional assumptions made for the conceptual model are: (1) the aquitard is homogeneous, anisotropic and of infinite lateral extent with finite-thickness; (2) the direction of flow in the aquitard is only vertical. It is worthwhile to point out that the well skin and non-uniform radial gradient along the pumping well screen are not taken into account for the sake of model simplicity. The boundary condition at the well face for a partially penetrating well is usually treated in two ways. On way is to assume a uniform radial flux along the well screen in analytical derivation (e.g. Hantush, 1961; Neuman, 1974; Lee, 1999; Yang et al., 2006; Mishra et al., 2012 and others) which is used in this study, the other way allows for non-uniform radial flux distribution along the well screen (Cassiani and Kabala, 1998; Hemker, 1999a, 1999b; Chang and Chen, 2003; Perina and Lee, 2006). The latter usually adopts the discretization method to divide the screen length into several segments, and involves laboring (but straightforward) mathematical derivations. The impact of well skin and non-uniform radial flux will be further addressed in the discussion section later.

The coordinate system is set up as follows. The origin is at the center of the pumping well and is located at the bottom of the aquifer with the r and z axes along the outward radial direction and upward vertical direction, respectively (L). The values of the aquifer and the aquitard thickness are B and β, respectively (L). l and d are the upper and lower vertical coordinates of the well screen, respectively (L). The well radii in the aquifer and above the aquifer (casing) could be different and are r_w and r_c, respectively (L).

2.1. Transient flow solution

According to the above conceptual model, the governing equation, which describes the drawdown distribution, s(r,z,t), in the domain r_w ≤ r ≤ ∞ and 0 ≤ z ≤ B for axisymmetric flow to a pumped well in an anisotropic leaky confined aquifer, can be expressed as:

\[ K_r \frac{\partial^2 s(r,z,t)}{\partial r^2} + \frac{K_r}{r} \frac{\partial s(r,z,t)}{\partial r} + K_z \frac{\partial^2 s(r,z,t)}{\partial z^2} = S_s \frac{\partial s(r,z,t)}{\partial t}. \]  

(1)

where \( K_r \) and \( K_z \) respect the radial and vertical hydraulic conductivities, respectively (L/T); \( S_s \) is the aquifer specific storage (1/L) and \( t \) is the pumping time (T).

The drawdown is initially zero:

\[ s(r,z,0) = 0. \]  

(2)

The drawdown at the infinite lateral boundary for the main aquifer is zero as well:

\[ s(r,B) = 0 \]  

(3)
\( s(\infty, z, t) = 0. \)  

Considering the wellbore storage, the boundary condition at the wellbore with uniform radial flux for a partially penetrating well can be written as:

\[
Q = -2\pi K_r r_w(l - d) \frac{\partial s(r_w, z, t)}{\partial r} + C_w \frac{\partial s(r_w, z, t)}{\partial t}, \quad d \leq z \leq l, \tag{4}
\]

And the boundary condition along the well casing is:

\[
\frac{\partial s(r_w, z, t)}{\partial r} = 0, \quad 0 < z < d, \quad l < z < B, \tag{5}
\]

where \( Q \) is the pumping rate (L\(^2\)/T); \( C_w \) is the wellbore storage coefficient (L\(^3\)), equal to \( \pi r_w^2 \).

The bottom aquifer boundary condition is:

\[
\frac{\partial s(r, 0, t)}{\partial z} = 0. \tag{6}
\]

The continuity of flux boundary at aquitard–aquitiver interface is:

\[
K_z \frac{\partial s(r, z, t)}{\partial z} = K_z \frac{\partial s'(r, z, t)}{\partial z}, \quad z = B, \tag{7}
\]

in which \( K' \) is the aquitard vertical hydraulic conductivity (L/T); \( s'(r, z, t) \) is the drawdown in the aquitard (L).

The continuity of drawdown boundary at aquitard–aquitiver interface is:

\[
s'(r, z, t) = s(r, z, t), \quad z = B. \tag{8}
\]

The governing flow equation in the aquitard is:

\[
K'_z \frac{\partial^2 s'(r, z, t)}{\partial z^2} = S' \frac{\partial s'(r, z, t)}{\partial t}, \quad B \leq z \leq B + B'. \tag{9}
\]

The initial condition of drawdown in the aquitard is:

\[
s'(r, z, 0) = 0. \tag{10}
\]

The boundary condition on the top of the aquitard is:

\[
s'(r, z, t) = 0, \quad z = B + B'. \tag{11}
\]

Defining the dimensionless variables shown in Table 1, and applying the Laplace transform to Eqs. 1–11, the results will be:

\[
\frac{\partial^2 \tilde{s}_D(r_0, z_0, p)}{\partial r_0^2} + \frac{1}{r_0} \frac{\partial \tilde{s}_D(r_0, z_0, p)}{\partial r_0} + K_D \frac{\partial^2 \tilde{s}_D(r_0, z_0, p)}{\partial z_0^2} = K_0 \tilde{S}_D(r_0, z_0, p), \tag{12}
\]

where

\[
\tilde{S}_D(r_0, z_0, p) = \sum_{n=0}^{\infty} \left[ \frac{2\delta_1 K_0 q_n r_w}{p(l_0 - d_0)} \right] \left[ C_{wD} K_D q_n K_0 (q_n r_w) \delta_2 (l_0 - d_0) + q_n r_w K_1 (q_n r_w) \delta_1 \right], \tag{23}
\]
in which \( I_0(\cdot) \) and \( I_1(\cdot) \) are respectively the first kind of the modified Bessel functions with order of zero and one; \( K_0(\cdot) \) and \( K_1(\cdot) \) are respectively the second kind of the modified Bessel functions with order of zero and one.

Simultaneously, the drawdown in the aquitard in Laplace domain can be obtained by substituting Eq. (23) into (21), the result is:

\[
\ddot{s}_D(t_0, z_0, p) = \frac{\sinh(\sqrt{2p}(1 + B_0 - z_0))}{\sinh(\sqrt{2p}B_0)} \left\{ \sum_{n=0}^{\infty} p|l_0 - d|C_{0p}K_0(q_p z_0) \cos(\alpha_n) \right\} + \frac{2\delta_1 K_0(q_p t_0) \cos(\alpha_n)}{l_0 - d_0 + \frac{q_p z_0 K_1(q_p z_0)\delta_3}{2}} \right\}
\]

Because \( \alpha_n \) is associated with \( p \) in Eqs. (23) and (24), it is not easy to analytically invert the solutions in Laplace domain. Fortunately, there are many numerical inverse Laplace transform methods such as Stehfest (1970), Talbot (1979), de Hoog et al. (1982) to meet the need. Among them, the Stehfest method (Stehfest, 1970) has been successfully applied in similar studies such as Moench (1985, 1997), Sun and Zhan (2006), Zhan and Bian (2006) and Pasandi et al. (2008) for flow in leaky confined aquifers. In this study, the method of Stehfest (1970) is also employed to obtain the solutions in the time domain.

Notably, under the specific condition of no leakage from the aquitard, the newly derived solution in Laplace domain collapses to that of Yang et al. (2006, Eq. (7)) without considering the wellbore storage, and to that of Papadopoulos and Cooper (1967, Eq. (10)) when the pumping well is fully penetrating. The new solution is also in good agreement with the solution of Hunt (2005, Eq. (23)) without accounting for the effect of aquitard storage, well radius and wellbore storage.

2.2. Steady-state flow solution

When the pumping time is sufficiently long or the aquitard is relatively thin in thickness, the transient flow might reach the steady-state. The steady-state solution can be obtained from the final value theorem (Churchill, 1972) \( \lim_{t \to \infty} \ddot{s}_D(t_0, z_0, t_0) \to \infty \) where \( \ddot{s}_D(t_0, z_0) \) is the steady-state approximation of \( \ddot{s}_D(t_0, z_0) \). On the basis of Eq. (23), the steady-state drawdown in the aquifer in the time domain is expressed as:

\[
\ddot{s}_D(t_0, z_0, t_0 \to \infty) = \sum_{n=0}^{\infty} \frac{8K_0(\sqrt{p} \alpha_n r_0) \cos(\alpha_n z_0) \sin(\alpha_n l_0) - \sin(\alpha_n d_0)}{l_0 - d_0 |K_0(\alpha_n r_0) r_0 K_1(\sqrt{p} \alpha_n r_0) [2\alpha_n + \sin(2\alpha_n)]}.
\]

The steady-state drawdown in the aquitard is obtained from Eq. (24) as:

\[
\ddot{s}_D(t_0, z_0, t_0 \to \infty) = \sum_{n=0}^{\infty} \frac{8(1 + z_0 - \alpha_n r_0) \cos(\alpha_n l_0) \sin(\alpha_n d_0) - \sin(\alpha_n d_0)}{B_0(l_0 - d_0) |K_0(\alpha_n r_0) d_0 K_1(\alpha_n r_0) [2\alpha_n + \sin(2\alpha_n)]}.
\]

3. Flow at the aquitard–aquifer interface

In most investigations involving pumping wells, drawdown is usually a key concern, but the induced leakage rate and volume at the aquitard–aquifer interface are probably also equally important for pumping in a leaky confined aquifer. This may be understandable in several aspects. Firstly, the leakage across the aquitard–aquifer interface represents an important water supply for the aquifer and it will affect the drawdown distribution in the aquifer as well, it is somewhat similar to the "recharge" received by the aquifer; on the other hand, such a leakage represents the "loss" of water from the aquitard, and it may even result in the compression of the weak aquitard materials and leads to the so-called land subsidence. Secondly, the chemistry and the age of the pore water in the aquitard could be quite different from their counterparts of the aquifer, thus the induced leakage across the aquitard–aquifer interface will result in the mixing of pore waters from the aquitard and aquifer near such an interface. Such a mixing process is of great concern in terms of water quality management and aquifer age dating (Lawrence et al., 2000; McMahon, 2001; Bethke and Johnson, 2002a, 2002b).

For instance, McMahon (2001) revealed that important biogeochemical reactions such as O₂ reduction, denitrification, and Fe³⁺, SO₂⁻ and CO₂ reduction can occur near the aquitard–aquifer interface, which represented a mixing zone capable of supporting greater microbial activity than either the aquifer or the aquitard alone. Lawrence et al. (2000) used a case study at Hat Yai of Thailand to demonstrate that downward leakage of shallow groundwater, which has been contaminated by urban wastewaters and naturally occurred arsenic, through the upper aquitard to the deep semi-confined aquifer, caused long-term water quality problems for the deep aquifer. Bethke and Johnson (2002a, 2002b) showed that the contribution of aquitards to the age of groundwater in aquifers closely depended on the ratios of fluid volumes in aquitards to aquifers.

The rate and volume of leakage will be obtained in the following. Firstly, the definition of the leakage rate (\( \Gamma \)) can be expressed as:

\[
\Gamma(R, t) = \int_0^R 2\pi q'dr.
\]

in which \( q' = K' \partial \dot{s} / \partial R \), is the vertical specific discharge at the aquitard–aquifer interface; \( R \) is the arbitrary radial distance from the pumping well.

The definition of the leakage volume (\( V \)) within a given distance and the arbitrary pumping time \( t \) can be written as:

\[
V(R, t) = \int_0^t \Gamma(R, \tau)d\tau.
\]

Using the dimensionless parameters defined in Table 1, Eqs. (28) and (29) can be rewritten as:

\[
\Gamma_{bD}(R_0, t_0) = \frac{1}{2} \int_0^{R_0} r_0 K_0'(\partial \dot{\sigma}_D / \partial R_0) dr_0,
\]

\[
V_{bD}(R_0, t_0) = \int_0^{t_0} \Gamma_{bD}(R_0, \tau)d\tau.
\]

Applying Laplace transform to Eq. (30) and substituting Eq. (23) into it, one can obtain the dimensionless leakage rate in Laplace domain as:

\[
\text{In which } K_0(\cdot) \text{ and } K_1(\cdot) \text{ are respectively the second kind of the modified Bessel functions with order of zero and one.}
\]


77
\[
\Gamma_D(R_0, p) = -\sum_{n=0}^{\infty} \int_0^R 2\delta_1 \omega_n K_D(C_{ud} K_D p K_0(q_{r^0} R_0))^2 (\frac{1}{Q_0} - \frac{1}{Q_1}) \sin(\omega_n \Delta q) \, dr_0.
\] (32)

Notably, \(\Gamma_D\) is always negative, meaning that flow at the aquitard–aquifer interface is downward. The integration of the modified Bessel function \(K_0(\cdot)\) in Eq. (32) can be calculated analytically using the identity \(\int_0^R t^n e^{\lambda t} \, dt = x^n e^{\lambda x} \Gamma(n+1, \lambda)\) of Spanier and Oldham (1987, p.506, 51:10:8). Accordingly, one has:

\[
\Gamma_D(R_0, p) = -\sum_{n=0}^{\infty} \frac{\delta_1 \omega_n K_D [1 - q_{r^0} R_0 K_1(q_{r^0} R_0)] \sin(\omega_n)}{C_{ud} K_D p K_0(q_{r^0} R_0)^2 \Delta q / (l_0 - d_0) + q_{r^0} R_0 K_1(q_{r^0} R_0) \Delta q}.
\] (33)

Because \(q_r R_0 K_1(q_r R_0)\) at \(R_0 = 0\) is always negative, meaning that flow at the aquifer–aquitard interface is downward. The total leakage rate through the whole aquitard–aquifer interface in Laplace domain as:

\[
\Gamma_D(R_0, p) = -\sum_{n=0}^{\infty} \frac{\delta_1 \omega_n K_D \sin(\omega_n)}{C_{ud} K_D p K_0(q_{r^0} R_0)^2 \Delta q / (l_0 - d_0) + q_{r^0} R_0 K_1(q_{r^0} R_0) \Delta q}.
\] (34)

Similarly, the steady-state dimensionless leakage rate in the time domain can be expressed as:

\[
\Gamma_D(R_0, t_0 \rightarrow \infty) = \lim_{p \rightarrow 0} \left[\frac{\Gamma_D(R_0, p)}{p}\right] = -\sum_{n=0}^{\infty} \frac{4 \sin(\Delta q) [1 - \sqrt{K_D} e_0 R_0 K_1(\sqrt{K_D} e_0 R_0)] [\sin(e_0 l_0) - \sin(e_0 d_0)]}{\sqrt{K_D} e_0^2 R_0 K_1(\sqrt{K_D} e_0 R_0) [2 e_0^2 + \sin(2 e_0)]}.
\] (35)

When \(R_0 \rightarrow \infty\), one obtains the steady-state dimensionless leakage rate through the whole aquitard–aquifer interface in real-time domain as:

\[
\Gamma_D(R_0 \rightarrow \infty, t_0 \rightarrow \infty) = -\sum_{n=0}^{\infty} \frac{4 \sin(\Delta q) [\sin(e_0 l_0) - \sin(e_0 d_0)]}{\sqrt{K_D} e_0^2 R_0 K_1(\sqrt{K_D} e_0 R_0) [2 e_0^2 + \sin(2 e_0)]}.
\] (36)

The leakage volume within an arbitrary distance in Laplace domain can be expressed as (Zhan and Bian, 2006):

\[
V_0(R_0, p) = \Gamma_D(R_0, p) / p.
\] (37)

The total leakage volume through the whole aquitard–aquifer interface in Laplace domain is expressed as:

\[
V_0(R_0, p) = \Gamma_D(R_0 \rightarrow \infty, p) / p.
\] (38)

Applying the final value theorem (Churchill, 1972) and following the method provided by Zhan and Bian (2006), the steady-state dimensionless leakage volume, \(V_0(R_0 \rightarrow \infty)\), can be obtained through multiplying \(t_0\) by \(\Gamma_D\) shown in Eq. (36). Similarly, simply multiply \(t_0\) by \(\Gamma_D\) shown in Eq. (36), the steady-state dimensionless total leakage volume through the whole aquitard–aquifer boundary, \(V_0(R_0 \rightarrow \infty, t_0 \rightarrow \infty)\), is able to be obtained.

\[K' = 10^{-8} \text{ m/s}; B = 25 \text{ m}; S_1' = 10^{-3} \text{ m}^{-1}; Q = 10^{-3} \text{ m}^3/\text{s}; K_2 = 10^{-5} \text{ m/s}; K_v = 10^{-3} \text{ m/s}; S_v = 2 \times 10^{-3} \text{ m}^{-1}; l = 18.75 \text{ m} ; d = 6.25 \text{ m}; \]

\(z = 12.5 \text{ m} \). The choice of above default parameter values are similar to those used by Moench (1997) and Sun and Zhan (2006).

4. Results and sensitivity analysis

In the following, the drawdown is analyzed systematically. The parameter defaults used in all the following figures are given in Table 2 and they are: \(r_w = 0.05 \text{ m}; r_s = 0.25 \text{ m}; B = 5, 10, 20 \text{ m}; \)

4.1. Effect of aquifer or aquitard parameters on drawdown

Hunt (2005) obtained an analytical solution for flow to a partially penetrating well in a leaky confined aquifer neglecting the wellbore storage and the aquitard storage. Fig. 2 shows the dimensionless drawdown versus the dimensionless time of this study and those of Hunt (2005). Note that the case for present solution with ignoring the effect of wellbore storage is considered as well as a reference. One can find that the dimensionless drawdown calculated by the present solution without considering the effect of wellbore storage is in good agreement with the solution of Hunt (2005). When the effect of the aquitard storage is considered, the
drawdown induced by a pumping well of an infinitesimal radius with no wellbore storage agrees well with that of the present solution and smaller than that of Hunt (2005) at intermediate times. All the drawdown curves have the same value at late times, suggesting that the wellbore storage and the aquitard storage effects disappear eventually.

Since the vertical flow could be considerable for a partially penetrating well, especially near the pumping well at early and intermediate times, thus it is necessary to analyze the influence of the aquifer anisotropy on the drawdown. For the purpose of illustration, the well screen is always located at the middle part of the aquifer. Fig. 3 shows the dimensionless drawdown versus the dimensionless time for different aquifer anisotropic ratios (i.e., vertical/radial hydraulic conductivity ratios) \( K_D \). The value for \( K_D \) is chosen to be 0.001, 0.01, 0.1 or 1. Note that the aquifer is isotropic for \( K_D = 1 \). Obviously, all the curves approach their asymptotic values at late times, and the dimensionless drawdown decreases as \( K_D \) increases. A smaller \( K_D \) indicates greater difficulty to drain the water vertically. Hence greater amounts of water flow to the well from the radial direction and the larger drawdown will be found near the pumping well compared to the isotropic case.

Fig. 4 shows the dimensionless drawdown versus the aquitard–aquifer specific storage ratio \( c \) with \( B_D = 0.2, r_D = 0.1, r_w = 0.01, \phi = 0.5, z_D = 0.5, K_D = 0.1, \gamma = 500, \beta = 0.001, c_{aqd} = 0.1 \).

Fig. 5 shows the dimensionless drawdown versus the dimensionless time for different aquitard–aquifer thickness ratio \( B_D \) with \( r_D = 0.1, r_w = 0.01, \phi = 0.5, z_D = 0.5, K_D = 0.1, c_{aqd} = 0.1, \beta = 0.001, c_{aqd} = 0.1, \gamma = 500, \beta = 0.001, B_D = 0.002, 0.02, 0.2 \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Default value</th>
<th>Parameter</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>25 m</td>
<td>( B' )</td>
<td>5 m</td>
</tr>
<tr>
<td>( r_w )</td>
<td>0.25 m</td>
<td>( r_c )</td>
<td>0.25 m</td>
</tr>
<tr>
<td>( r )</td>
<td>2.5 m</td>
<td>( z )</td>
<td>12.5 m</td>
</tr>
<tr>
<td>( l )</td>
<td>18.75 m</td>
<td>( d )</td>
<td>6.25 m</td>
</tr>
<tr>
<td>( Q )</td>
<td>0.001 m²/s</td>
<td>( K_c )</td>
<td>1 × 10⁻⁴ m/s</td>
</tr>
<tr>
<td>( K' )</td>
<td>1 × 10⁻⁴ m/s</td>
<td>( K_s )</td>
<td>2 × 10⁻¹ m⁻¹</td>
</tr>
</tbody>
</table>

Note: When different aquitard/aquifer thickness ratios are used, the properties of the aquifer are fixed and only the corresponding aquitard parameters are changed.

### Table 2
The default values used in this study.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Default value</th>
<th>Parameter</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>25 m</td>
<td>( r_w )</td>
<td>0.25 m</td>
</tr>
<tr>
<td>( r )</td>
<td>2.5 m</td>
<td>( z )</td>
<td>12.5 m</td>
</tr>
<tr>
<td>( l )</td>
<td>18.75 m</td>
<td>( d )</td>
<td>6.25 m</td>
</tr>
<tr>
<td>( Q )</td>
<td>0.001 m²/s</td>
<td>( K_c )</td>
<td>1 × 10⁻⁴ m/s</td>
</tr>
<tr>
<td>( K' )</td>
<td>1 × 10⁻⁴ m/s</td>
<td>( K_s )</td>
<td>2 × 10⁻¹ m⁻¹</td>
</tr>
</tbody>
</table>

### Fig. 2
Comparison of the dimensionless time-drawdown behavior in this study and in Hunt (2005) with \( B_D = 0.2, r_D = 0.1, r_w = 0.01, \phi = 0.5, z_D = 0.5, K_D = 0.1, \gamma = 500, \beta = 0.001, c_{aqd} = 0.1 \).

### Fig. 3
The dimensionless drawdown versus the dimensionless time for different aquifer anisotropy with \( B_D = 0.2, r_D = 0.1, r_w = 0.01, \phi = 0.5, z_D = 0.5, K_D = 0.1, \gamma = 500, \beta = 0.001, c_{aqd} = 0.1 \).
implies a small increment, which is often chosen as 1. = 0.01, respectively the normalized sensitivity of the on drawdown sus-
þ at the intermediate and respects the dependent variable of the ¼ 0.1, = 0.1, = 0.0001, 0.001, 0.01.

The effect of the aquitard/aquifer thickness ratio (BD) on the dimensionless drawdown is illustrated in Fig. 5. The dimensionless time-drawdown curves shown in this figure are the same at early times and the drawdown increases with BD. A larger BD implies a relatively thicker aquitard and less leakage entering the aquifer, resulting in a larger drawdown at late times.

To explore the influence of different aquitard/aquifer vertical hydraulic conductivity ratio (b) on the response of the leaky confined aquifer, we generate type curves shown in Fig. 6. Because of water derived entirely from the wellbore storage at early times, the dimensionless drawdown and the dimensionless time is linearly proportional. However, at the intermediate and late times, a larger aquitard/aquifer hydraulic conductivity ratio leads to a smaller drawdown. A greater b means a faster release of the storage water within the aquitard and a greater rate of leakage through the aquitard. Meanwhile, a larger b leads to a short period of time for approaching the asymptotic behavior.

4.2. Flow at the aquitard–aquifer interface

From the perspective of groundwater management, the rate and volume of leakage at the aquitard–aquifer interface are of great concern. In this study, the dimensionless leakage rate at the aquitard–aquifer interface within a dimensionless radial distance R0 from the partially penetrating pumping well is able to be obtained by Eq. (33). The total leakage rate through the entire aquitard–aquifer interface for any given time can be evaluated by Eq. (34).

The dimensionless total leakage volume through the entire aquitard/aquifer interface shown in Fig. 7 with different aquitard/aquifer specific storage ratio r. The differences among curves for different r values in Fig. 7 are evident at the early and intermediate times, and no difference is seen at the late times. Fig. 7 also shows that there is a linear relationship between the dimensionless total leakage volume and the dimensionless time after a certain lapse of time, and a larger r leads to a longer time to approach the linear function. The slope of the linear segments is the same as the steady-state leakage rate.

4.3. Sensitivity analysis

In order to access the degree of sensitivity of the aquifer drawdown to the aquifer and aquitard parameters, a normalized sensitivity analysis method proposed by Yeh and Huang (2005) and Huang and Yeh (2007) has been done for CwD, KD, and the aquitard/aquifer thickness ratio (BD). The normalized sensitivity of a given parameter can be defined as:

\[ X_{ij} = P_j \frac{\partial O_i}{\partial P_j} , \]  \tag{39} 

in which \( X_{ij} \) respects the normalized sensitivity of the j-th parameter (Pj) at the i-th time and Oi respects the dependent variable of the model (drawdown). A finite difference scheme is used by Yeh and Huang (2005) and Huang and Yeh (2007) to approximate the partial derivative as:

\[ X_{ij} = P_j \frac{\partial O_i}{\partial P_j} = P_j \frac{O_i(P_j + \Delta P_j) - O_i(P_j)}{\Delta P_j} , \]  \tag{40} 

in which \( \Delta P_j \) is a small increment, which is often chosen as \( 10^{-2} \times P_j \). Fig. 8 shows the drawdown sensitivity to parameters CwD, KD, and the aquifer anisotropic ratio (BD) for a leaky confined aquifer with the base values of BD = 0.2, r0 = 0.1, rwd = 0.01, = 0.5, zd = 0.5, KD = 0.1, c = 0.1, and r = 0.001. The absolute values of \( X_{ij} \) are used in this figure for the purpose of comparison. The normalized drawdown sensitivities with respect to BD are positive and those with respect to the remaining parameters are negative.

One can make a few observations. First, the drawdown is most sensitive to the well partial penetration and the sensitivity approaches a constant at the late times. Secondly, the drawdown is apparently sensitive to the wellbore storage CwD and the aquifer anisotropic ratio KD at the early times and the degree of sensitivity to KD is higher than to CwD. And a relative change in CwD has negligible influence upon the drawdown at the late times [as expected], while the effect of KD on drawdown sustains over the entire time of pumping. Thirdly, the drawdown is moderately sensitive to the aquitard/aquifer specific storage ratio r at the intermediate times only; the drawdown is moderately sensitive to the aquitard/aquifer vertical hydraulic conductivity ratio b and the aquitard/aquifer thickness ratio BD at the intermediate and
and drawdown sensitivities in respect to later times. Fourthly, the drawdown sensitivities with respect to The absolute normalized sensitivities of the parameters Fig. 8. face flow with the minimized parameter numbers, the well skin and 

such parameters may also be difficult to determine from a practical non-negligible (e.g. Park and Zhan, 2002; Zhan and Park, 2003; Sun aging can be significant, and its influence upon flow in the aquifer is and Yeh, 1990; Carrera et al., 2005). Extensive discussions on the modeler to identify areas of the model where inverse procedures will be more subject to error (McElwee, 1982; Yeh, 1986; Sun (the early and intermediate times). The aquitard effect will be diminished at the late pumping time. It is important to quantitatively assess the spatial and temporal scales in which the involving parameters (such as the aquitard storage, the well skin, the non-uniform radial gradient along the pumping well screen) affect the drawdown behavior the most. This is also a subject that deserves a careful investigation in the future.

6. Summary and conclusions

A new semi-analytical solution for flow toward a partially penetrating constant-rate pumping well in an anisotropic leaky confined aquifer is developed by using Laplace transform and separation of variables. The solution is different from similar investigations by previous researchers in considering the jointed effects of aquitard storage, aquifer anisotropy, and wellbore storage by treating the aquitard leakage as a boundary condition at the aquitard–aquifer interface rather than a volumetric source/sink term in the governing equation. Moreover, Laplace-domain and steady-state solutions are obtained to calculate the rate and volume of (total) leakage through the aquitard–aquifer interface. And more importantly, the solutions are useful to manage similar aquifers for pumping induced flow occurs at the aquitard–aquifer interface. In addition, the steady-state leakage induced by the partially penetrating pumping well mainly depend on a few parameters such as the well screen length, the well radius, aquifer anisotropy and the vertical hydraulic conductivity of the aquitard and the thickness of the aquifer and aquitard.

The newly developed solution can be used to produce the time-drawdown curves for examining the effect of different aquifer and aquitard parameters on the drawdown. The results show that the wellbore storage and aquitard storage in many cases should not be neglected, particularly during the early and intermediate pumping stages. And the sensitivity analysis for drawdown reveals the following findings. The drawdown is most sensitive to the well partial penetration and the sensitivity approaches a constant at the late times. The drawdown is apparently sensitive to the wellbore storage and the aquifer anisotropic ratio at the early times and the degree of sensitivity to the aquifer anisotropic ratio is higher than to the well bore storage. The effect of the aquifer anisotropic ratio on drawdown sustains over the entire time of pumping. The drawdown is moderately sensitive to the aquitard/aquifer specific storage ratio at the intermediate times only; the drawdown is moderately sensitive to the aquitard/aquifer vertical hydraulic conductivity ratio and the aquitard/aquifer thickness ratio at the intermediate and later times. The sensitivities of drawdown with respect to the well partial penetration, the aquifer

anisotropic ratio, the aquitard/aquifer vertical hydraulic conductivity ratio and the aquitard/aquifer thickness ratio maintain constant values at late times. The drawdown sensitivities in respect to the aquitard/aquifer vertical hydraulic conductivity ratio and the aquitard/aquifer thickness ratio are the same at late times.

Appendix A

Assuming that the principle of separation of variable works and \( \delta_0 \) is the product of a function of \( r_0 \) and a function of \( z_0 \):

\[
F(r_0, p)G(z_0, p) = F(r_0, p)G(z_0, p), \quad \text{Eq. (12) can be transformed to:}
\]

\[
G \frac{\partial^2 F}{\partial r^2} + \frac{1}{r_0} \frac{\partial F}{\partial r} + Fk_0 \frac{\partial^2 G}{\partial z^2} = pk_0FG. \quad \text{(A1)}
\]

Eq. (A1) can be separated to get the following system of equations:

\[
\frac{\partial^2 G}{\partial z^2} + \alpha^2 G = 0. \quad \text{(A2)}
\]

\[
\frac{\partial^2 F}{\partial r^2} + \frac{1}{r_0} \frac{\partial F}{\partial r} - [K_0(p + \alpha^2)]F = 0. \quad \text{(A3)}
\]

The solution of Eq. (A2) subject to the boundary condition Eq. (16) is:

\[
G(z_0, p) = a_n(p) \cos(\omega_n z_0). \quad \text{(A4)}
\]

where \( a_n(p) \) is constant with respect to \( z_0 \). Substituting Eq. (A4) into Eq. (22) yields:

\[
\omega_n \tan(\omega_n) = \beta / \sqrt{pB_0}. \quad \text{(A5)}
\]

\( \omega_n \) is obtained by using the Newton–Raphson method to Eq. (A5) (Press et al., 1989). It is worthwhile to point out that Eq. (A5) is a multiple-root equation and the Newton–Raphson method needs to be supplied with a suitable initial guess for each root. Previous studies such as Moench (1997), Zhan and Zlotnik (2002), Hunt (2005), Park and Zhan (2002), Zhan and Park (2003), Perina and Lee (2006), and Sun and Zhan (2006) have explained the periodicity of the roots of the equation and provided the root-searching technique for this type of problems.

The solution of Eq. (A3) is:

\[
F(r_0, p) = b_n(p)b_0(q_r, r_0) + c_n(p)K_0(q_r, r_0), \quad \text{(A6)}
\]

where \( b_n(p) \) and \( c_n(p) \) are constants, and \( q_r = \sqrt{K_0(p + \alpha^2)} \).

The product of Eqs. (A4) and (A6) gives:

\[
\delta_0(r_0, z_0, p) = u_n(p)K_0(q_r, r_0) \cos(\omega_n z_0). \quad \text{(A7)}
\]

which where \( u_n(p) \) represents the product of \( a_n(p) \) and \( c_n(p) \). The complete solution for \( \delta_0 \) is to be obtained as:

\[
\delta_0(r_0, z_0, p) = \sum_{n=0}^{\infty} \delta_n(r_0, z_0, p) = \sum_{n=0}^{\infty} u_n(p)K_0(q_r, r_0) \cos(\omega_n z_0). \quad \text{(A8)}
\]

If the boundary condition Eq. (14) is employed, one has:

\[
\sum_{n=0}^{\infty} - q_nU_n(p)K_1(q_r r_0) \cos(\omega_n z_0) = \frac{C_0k_0 p}{r_0(1 - d_0)} \sum_{n=0}^{\infty} u_n(p)K_0(q_r r_0) \cos(\omega_n z_0)
\]

\[
- \frac{2}{r_0(1 - d_0)} \int_{a_0}^{b_0} \cos(\omega_n z_0) \cos(\omega_n z_0) \, dz_0. \quad \text{(A9)}
\]

Recalling Eq. (15), one obtains:

\[
\sum_{n=0}^{\infty} - q_nU_n(p)K_1(q_r r_0) \cos(\omega_n z_0) = 0. \quad \text{(A10)}
\]

Multiplying Eq. (A9) by \( \cos(\omega_n z_0) \) and integrating over the screen interval, one has:

\[
\sum_{n=0}^{\infty} - q_nU_n(p)K_1(q_r r_0) \int_{a_0}^{b_0} \cos(\omega_n z_0) \cos(\omega_n z_0) \, dz_0
\]

\[
= \frac{C_0k_0 p}{r_0(1 - d_0)} \sum_{n=0}^{\infty} u_n(p)K_0(q_r r_0) \int_{a_0}^{b_0} \cos(\omega_n z_0) \cos(\omega_n z_0) \, dz_0
\]

\[
- \frac{2}{r_0(1 - d_0)} \int_{a_0}^{b_0} \cos(\omega_n z_0) \, dz_0. \quad \text{(A11)}
\]

Multiplying Eq. (A10) by \( \cos(\omega_n z_0) \) and integrating over the sections below and above the screen respectively, one can obtain:

\[
\sum_{n=0}^{\infty} - q_nU_n(p)K_1(q_r r_0) \int_{a_0}^{b_0} \cos(\omega_n z_0) \cos(\omega_n z_0) \, dz_0 = 0, \quad \text{(A12)}
\]

and

\[
\sum_{n=0}^{\infty} - q_nU_n(p)K_1(q_r r_0) \int_{a_0}^{b_0} \cos(\omega_n z_0) \cos(\omega_n z_0) \, dz_0 = 0. \quad \text{(A13)}
\]

Adding Eqs. A11, A12, A13 together and considering the fact that the set \( \cos(\omega_n z_0) \) is orthogonal over the interval of \( z_0 \in [0, 1] \) (i.e., the only non-zero term is \( m = n \)), one obtains:

\[
- q_nU_n(p)K_1(q_r r_0) \int_{a_0}^{b_0} \cos^2(\omega_n z_0) \, dz_0
\]

\[
= \frac{C_0k_0 p}{r_0(1 - d_0)} \cdot u_n(p)K_0(q_r r_0) \int_{a_0}^{b_0} \cos^2(\omega_n z_0) \, dz_0
\]

\[
- \frac{2}{r_0(1 - d_0)} \int_{a_0}^{b_0} \cos(\omega_n z_0) \, dz_0. \quad \text{(A14)}
\]

From Eq. (A14), the constant can be solved as:

\[
u_n(p) = \frac{2}{p(1 - d_0) - \delta_1}
\]

\[
\times \frac{C_0k_0 p}{r_0(1 - d_0)} p_0K_0(q_r r_0) \delta_2/ \sqrt{pB_0} \cos(\omega_n z_0) \, dz_0. \quad \text{(A15)}
\]

where

\[
\delta_1 = \int_{a_0}^{b_0} \cos(\omega_n z_0) \, dz_0 = \frac{1}{\omega_n} \left( \sin(\omega_n a_0) - \sin(\omega_n b_0) \right). \quad \text{(A16)}
\]

\[
\delta_2 = \int_{a_0}^{b_0} \cos^2(\omega_n z_0) \, dz_0
\]

\[
= \frac{b_0 - d_0}{2} + \frac{1}{4\omega_n} \left( \sin(2\omega_n a_0) - \sin(2\omega_n b_0) \right). \quad \text{(A17)}
\]

\[
\delta_3 = \int_{a_0}^{b_0} \cos^2(\omega_n z_0) \, dz_0 = \frac{1}{2} \left( 1 + \frac{1}{2\omega_n} \sin(2\omega_n a_0) \right). \quad \text{(A18)}
\]

Consequently, the Laplace domain solutions of the drawdown can be obtained by substituting Eq. (A15) into Eq. (A7).

References


